Yet Another Derivation of the Mortgage Payment Equation

by James Bainter © 2013

This document shows how to derive the monthly payment for a mortgage used to finance houses and automobiles. The main characteristic of this type of mortgage is that a fixed (constant) amount of money is paid each month until the loan (principal) has been paid in full. Each monthly payment also includes an interest payment. This derivation will only consider the case were the interest rate is constant over the term of the mortgage.

Symbols used in derivation:
P = the principal amount of the mortgage
i = the monthly interest rate
n = number of mortgage payments
b = balance of mortgage (loan) after a payment
M = the monthly payment amount

M is make up of two terms - equity payment, and interest payment.
$M = e_n + i_n$
As the payments are made, the equity term (e_n) increases and the interest term (i_n) decreases keeping M constant.

Equity terms are calculated by subtracting the corresponding interest term from the monthly payment (M).
$e_n = M - b_{n-1}i$
The sum of the equity payments equals the (P) principal.
$P = e_1 + e_2 + e_3 + ... + e_n$

1st payment:
$e_1 = M - Pi$
$b_1 = P - M + Pi = P(1+i) - M$

2nd payment:
$e_2 = M - b_1 i = M - (P(1 + i) - M)i = M(1+i) - Pi(1+i) = (M-Pi)(1+i)$
$b_2 = b_1 - e_2 = P(1+i) - M - (M-Pi)(1+i) = (P+M-Pi)(1+i) - M$
$\quad = M(1+i) + P(1+i)^2 - M$

3rd payment:
$e_3 = M - b_2 i = M - (M(1+i) + P(1+i)2 - M)i$
$= M + Mi(1+i) Pi(1+i)^2 - Mi = M(1+i) + Mi(1+i) + Pi(1+i)^2$
$= M (1+i)^2 + Pi(1+i)^2 = (M+Pi)(1+i)^2$

The equations for e_2 and e_3 show that the solution to this problem involves a geometric series. See http://en.wikipedia.org/wiki/Geometric_series for information on geometric series.

If the sum of the equity payments is an geometric series a common ratio should exist between the terms of the sum.

Check first ratio:
$e_2/e_1 = (M-Pi)(1+i)/(M-Pi) = 1+i$

Check second ratio:
$e_3/e_2 = (M+Pi)(1+i)^2/((M-Pi)(1+i)) = 1+i$

The equity series is a geometric series with a common ratio of (1+i).
$P = (M-Pi) (1 + (1+i) + (1+i)^2 + (1+i)^3 ... + (1+i)^{n-2} + (1+i)^{n-1})$
From the above reference the sum of the series = $a (1-r)^n / (1-r)$
where first equity payment = a = (M-Pi) and r = (1+i)
$P = (M-Pi) (1-(1+i)^n) / (1-(1+i))$
$-Pi = (M-Pi) (1-(1+i)^n)$
$(M-Pi) = -Pi/(1-(1+i)^n)$

Solve for M the final solution:
$M = Pi (1 - 1/(1 - (1+i)^n))$

Test Example:

P = 100000 $100,000 loan
i = 5.0/1200   5% interest rate per year
n = 360   30 yr (360 months)

M = $536.82   payment per month

Total Interest paid over term of mortgage = nM-P = 360*536.82-100000
= $93,255.20
I suppose most mortgage payer's do not want to know this!

During the life of the mortgage a point may be reached where the amount of the equity payment will equal the interest portion of the monthly payment M. This "cross-over" point is derived as follows:

$e_p = i_p$
$M = e_p + e_p$; $M = 2e_p$
$e_p = M/2$
$M/2 = (M\text{-}Pi)(1+i)^{n-1}$
$(1+i)^{n-1} = M/(2(M\text{-}Pi))$
$(n-1)\log(1+i) = \log(M/(2(M\text{-}Pi)))$
Cross-over point is:
$n = \log(M/(2(M\text{-}Pi))) / \log(1+i) + 1$

For the Test Example:
n = 194.3 months or 16.2 years
Not all mortgages have a crossover point. If the lenght of the mortgage is short the the equity portion of the mortgage will always exceed the interest portion.

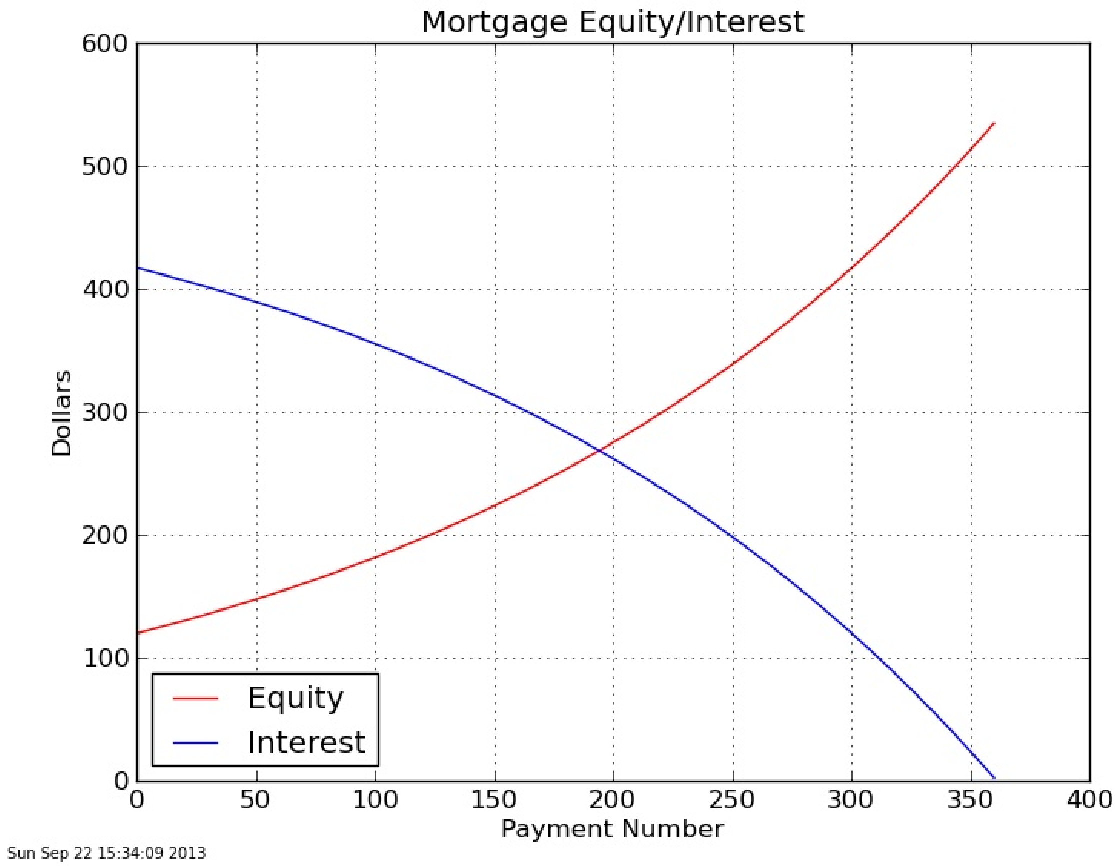

Figure 1

Figure 1 shows the "Equity" part of the mortgage payment M increasing and the "Interest" part decreasing as the mortgage payments are made. Note the Equity and Interest curves intersect at the crossover point computed at payment number 194.

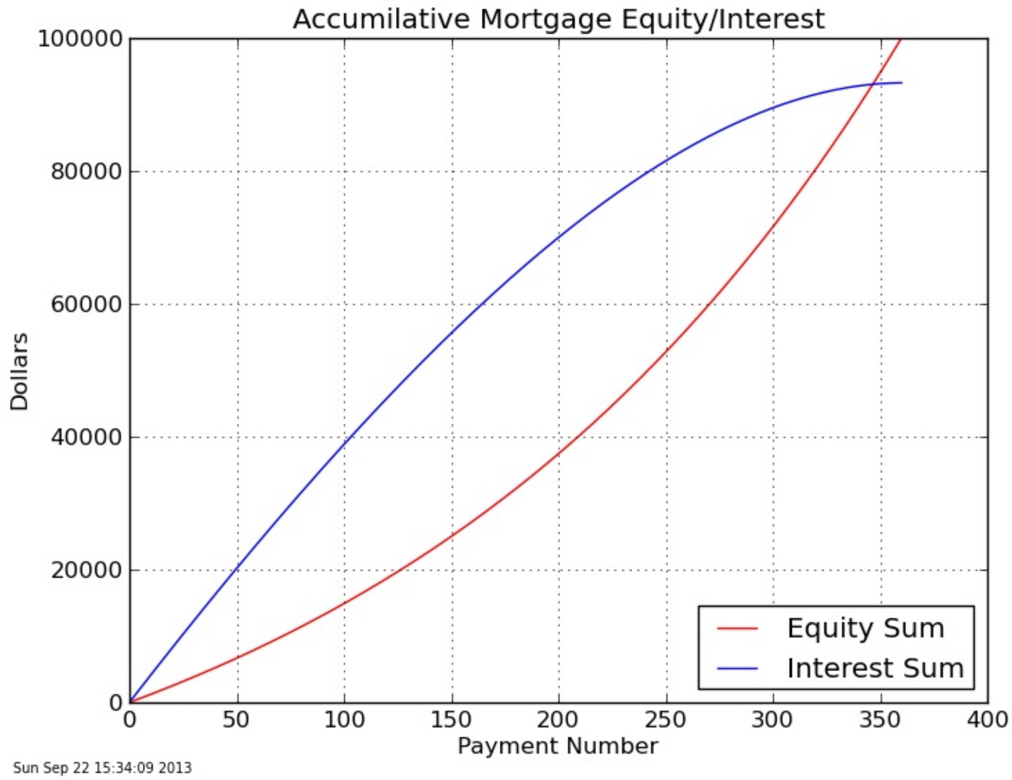

Figure 2

Figure 2 shows the accumilative equity and interest paid as the mortgage payments are made.